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On upper bounds for the spectral variation of two regular matrix pairs

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Abstract

Upper bounds for the spectral variation of two regular matrix pairs have been given in [Guoxing Wu, Optimal bounds for the spectral variation of two regular matrix pairs, Linear Algebra Appl. 418 (2006) 891–899; G.W. Stewart, An Elsner-like perturbation theorem for generalized eigenvalues, Linear Algebra Appl. 390 (2004) 1–5; Ren-cang Li, On the variation of the spectra of matrix pencils, Linear Algebra Appl. 139 (1990) 147–164]. In this note we show that some of the upper bounds are optimal and some others are strict.

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For the generalized eigenvalue problem, several optimal bounds for $S_Z(W)$ (see (1) for definition) have been determined in [1]. Are there any other optimal bounds for $S_Z(W)$? The purpose of this note is to show that some of the upper bounds in [1–3] are optimal and some others are strict.

Let $\mathbb{C}^{n\times n}$ denote the set of all $n \times n$ complex matrices. For $A, B \in \mathbb{C}^{n\times n}$, we call $Z = (A, B)$ a regular matrix pair if $\det(A - \lambda B) \not\equiv 0, \lambda \in \mathbb{C}$. We use $Z = (A, B)$ and $W = (C, D)$ for two regular matrix pairs with eigenvalues (α_i, β_i) and (γ_i, δ_i) respectively. The spectral variation of W with respect to Z is defined by

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$$S_Z(W) = \max_i \min_j \frac{|\alpha_j \delta_i - \beta_j \gamma_i|}{\sqrt{|\alpha_j|^2 + |\beta_j|^2}\sqrt{|\gamma_i|^2 + |\delta_i|^2}}. \tag{1}$$

We use $\| \ \|$ for both the Euclidean vector norm and the spectral norm. A^H stands for the conjugate transpose of A. We will use the metric for Z and W

$$d_2(Z, W) = \|Z^H(ZZ^H)^{-1}Z - W^H(WW^H)^{-1}W\|.$$

For $d_2(Z, W)$, Elsner and Sun [5] have shown that

$$d_2(Z, W) = \min\{\|(ZZ^H)^{-\frac{1}{2}}Z - KW\|, K \in C^{n\times n}\}. \tag{2}$$

By the generalized Schur decomposition [4], there are unitary matrices U_1 and U_2 such that

$$U_1^H AU_2 = R \quad \text{and} \quad U_1^H BU_2 = T, \tag{3}$$

where R and T are upper triangular. (r_{ii}, t_{ii}) are the eigenvalues of Z, which can be made to appear anywhere on the diagonals of R and T.

Let $D(Z) = \left(\max_{|\alpha|^2+|\beta|^2=1} |\det(\beta A - \alpha B)|\right)^{\frac{1}{n}}$, $\gamma(Z) = \max_{|\alpha|^2+|\beta|^2=1} \tilde{\sigma}_{\min}(\beta A - \alpha B)$, $\sigma(Z) = \sup_{\{U_1,U_2\}\in\mathscr{U}_{(Z)}} \min_{1\leqslant i\leqslant n} \sqrt{|r_{ii}|^2 + |t_{ii}|^2}$, where $\tilde{\sigma}_{\min}(X)$ denotes the smallest singular value of X and $\mathscr{U}_{(Z)} = \{\{U_1, U_2\}|\{U_1, U_2\}$ are all the unitary matrix pairs satisfying the equations (3) $\}$. From the definition of $\sigma(Z)$ and compactness of $\mathscr{U}_{(Z)}$, we know that there exists a decomposition (3) such that

$$\sigma(Z) = \min_{1\leqslant i\leqslant n} \sqrt{|r_{ii}|^2 + |t_{ii}|^2}. \tag{4}$$

Hereafter, assume that the decomposition (3) satisfies (4). We have

$$\begin{aligned} D(Z) &\leqslant \left(\prod_{i=1}^{n}(|r_{ii}|^2 + |t_{ii}|^2)\right)^{\frac{1}{2n}}, \\ \gamma(Z) &\leqslant \left(\prod_{i=1}^{n}(|r_{ii}|^2 + |t_{ii}|^2)\right)^{\frac{1}{2n}}, \\ \sigma(Z) &\leqslant \left(\prod_{i=1}^{n}(|r_{ii}|^2 + |t_{ii}|^2)\right)^{\frac{1}{2n}}. \end{aligned} \tag{5}$$

The following inequalities have been obtained in [1–3]

$$S_Z(W) \leqslant \frac{1}{D(Z)}\|Z\| d_2(Z, W)^{\frac{1}{n}}, \tag{6}$$

$$S_Z(W) \leqslant \frac{1}{\gamma(Z)}(\|A\|^2 + \|B\|^2)^{\frac{1}{2}(1-\frac{1}{n})}(\|A - C\|^2 + \|B - D\|^2)^{\frac{1}{2n}}, \tag{7}$$

$$S_Z(W) \leqslant \frac{1}{\sigma(Z)}\|Z\|^{1-\frac{1}{n}}\|Z - W\|^{\frac{1}{n}}. \tag{8}$$

In this note, first we show that the upper bound in (6) is optimal and we describe the set of regular matrix pairs for which the bound is attained. Then we show that the inequalities (7) and (8) are strict.

Theorem 1. *For $Z \neq W$, the following are equivalent*:

(i) $S_Z(W) = \frac{1}{D(Z)}\|Z\|d_2(Z, W)^{\frac{1}{n}}$;
(ii) $S_Z(W) = \frac{1}{\gamma(Z)}\|Z\|d_2(Z, W)^{\frac{1}{n}}$;
(iii) $S_Z(W) = \frac{1}{\sigma(Z)}\|Z\|d_2(Z, W)^{\frac{1}{n}}$;
(iv) $\exists\alpha, \beta \in \mathbb{C}$, *such that* $Z = (\alpha U, \beta U)$ *and* $(-\bar{\beta}, \bar{\alpha})$ *is an eigenvalue of* W, *where* U *is a unitary matrix*;
(v) $\exists\varepsilon \in \mathbb{C}, |\varepsilon| = 1$ *such that* $A = \varepsilon\|A\|U$, $B = \|B\|U$, $(-\varepsilon\|B\|, \|A\|)$ *is an eigenvalue of* W, *where* U *is a unitary matrix.*

Proof. We proceed by showing that (i) is equivalent to (iv), (ii) is equivalent to (iv), (iii) is equivalent to (iv), and (iv) is equivalent to (v).

Obviously (iv) implies (i); we shall show that (i) implies (iv). Assume that (i) is satisfied. Let (γ, δ) be such that

$$S_Z(W) = \min_j \frac{|r_{jj}\delta - t_{jj}\gamma|}{\sqrt{|r_{jj}|^2 + |t_{jj}|^2}}, \tag{9}$$

where (γ, δ) is an eigenvalue of W satisfying $|\gamma|^2 + |\delta|^2 = 1$.

Let $(v_1, v_2, \ldots, v_n)$ be a unitary matrix satisfying $\delta C v_1 = \gamma D v_1$. We have (see 2.12 of [1])

$$\left\| Z \begin{pmatrix} \delta v_1 \\ -\gamma v_1 \end{pmatrix} \right\| \leqslant \|Z\|\ \|((ZZ^H)^{-\frac{1}{2}} Z - KW)\|, \tag{10}$$

where $K \in C^{n\times n}$.

Combining (9) with (5), (10) and (2) gives

$$\begin{aligned} S_Z(W)^n &\leqslant \prod_{j=1}^{n} \frac{|r_{jj}\delta - t_{jj}\gamma|}{\sqrt{|r_{jj}|^2 + |t_{jj}|^2}} \\ &\leqslant \frac{|\det(\delta A - \gamma B)(v_1, v_2, \ldots, v_n)|}{D(Z)^n} \\ &\leqslant \frac{1}{D(Z)^n} \left\| Z \begin{pmatrix} \delta v_1 \\ -\gamma v_1 \end{pmatrix} \right\| \left\| Z \begin{pmatrix} \delta v_2 \\ -\gamma v_2 \end{pmatrix} \right\| \cdots \left\| Z \begin{pmatrix} \delta v_n \\ -\gamma v_n \end{pmatrix} \right\| \\ &\quad \text{(Hadamard's inequality)} \\ &\leqslant \frac{1}{D(Z)^n} \|Z\|^n d_2(Z, W) \end{aligned} \tag{11}$$

Equality in Hadamard's inequality holds only for orthogonal columns. There exist orthogonal and normalized vectors $u_1, u_2, \ldots, u_n$, such that

$$Z \begin{pmatrix} \delta v_1 \\ -\gamma v_1 \end{pmatrix} = \|Z\|d_2(Z, W)u_1,\ Z \begin{pmatrix} \delta v_2 \\ -\gamma v_2 \end{pmatrix} = \|Z\|u_2, \ldots, Z \begin{pmatrix} \delta v_n \\ -\gamma v_n \end{pmatrix} = \|Z\|u_n$$

or

$$\begin{aligned} &\gamma(u_1, \ldots, u_n)^H A(v_1, \ldots, v_n) - \delta(u_1, \ldots, u_n)^H B(v_1, \ldots, v_n) \\ &\quad = \operatorname{diag}(\|Z\|d_2(Z, W), \|Z\|, \ldots, \|Z\|). \end{aligned}$$

Let

$$a_{ij} = u_i^H A v_j,\ b_{ij} = u_i^H B v_j\ (i, j = 1, \ldots, n),$$
$$\widetilde{Z} = ((u_1, \ldots, u_n)^H A(v_1, \ldots, v_n), (u_1, \ldots, u_n)^H B(v_1, \ldots, v_n)).$$

By Cauchy–Schwarz inequality, we have

$$\sum_{j=1}^{n}(|a_{ij}|^2 + |b_{ij}|^2)^{\frac{1}{2}} \leqslant \|Z\|$$
$$= \|\widetilde{Z}\| = \|(a_{ij}), (b_{ij})\| \leqslant (|a_{ii}|^2 + |b_{ii}|^2)^{\frac{1}{2}}, \quad i = 2, \ldots, n. \tag{12}$$

Thus $a_{ij} = b_{ij} = 0(j \neq i), \delta a_{ii} - \gamma b_{ii} = \|Z\| = (|a_{ii}|^2 + |b_{ii}|^2)^{\frac{1}{2}},\ i = 2, \ldots, n.$

Equalities in Cauchy–Schwarz inequality (12) hold if and only if (a_{ii}, b_{ii}) and $(\bar{\delta}, -\bar{\gamma})$ are linearly dependent on each other, i.e., $(a_{ii}, b_{ii}) = \|Z\|(\bar{\delta}, -\bar{\gamma}) i = 2, \ldots, n$. (a_{ii}, b_{ii}) are now the eigenvalues of Z.

It follows from (11), (12) that

$\delta a_{11} - \gamma b_{11} = \|Z\| = (|a_{11}|^2 + |b_{11}|^2)^{\frac{1}{2}}, a_{1i} = b_{1i} = 0, i \neq 1$ and $(a_{11}, b_{11}) = \|Z\|(\bar{\delta}, -\bar{\gamma})$.

Write $\alpha = \|Z\|\bar{\delta}, \beta = -\|Z\|\bar{\gamma}, (u_1, \ldots, u_n)(v_1, \ldots, v_n)^H = U$, then $A = \alpha U, B = \beta U$ and $(\gamma, \delta) = \frac{1}{\|Z\|}(-\bar{\beta}, \bar{\alpha})$, i.e., $(-\bar{\beta}, \bar{\alpha})$ is an eigenvalue of W. Thus (i) implies (iv).

Similarly, by (5) one can show that (ii) is equivalent to (iv), and (iii) is equivalent to (iv).

For the equivalence of (iv) and (v), obviously (v) implies (iv). To verify that (iv) implies (v) if $A = 0$ or $B = 0$, there is nothing to prove, so assume $A \neq 0$ and $B \neq 0$, write $\varepsilon = \frac{\alpha\bar{\beta}}{|\alpha\beta|}$, and then (iv) implies (v). This completes the proof. □

But the following inequalities are strict.

Theorem 2. *For $Z \neq W$, then*

(i) $S_Z(W) < \frac{1}{D(Z)}\|Z\|^{1-\frac{1}{n}}\|Z - W\|^{\frac{1}{n}}$.
(ii) $S_Z(W) < \frac{1}{\gamma(Z)}\|Z\|^{1-\frac{1}{n}}\|Z - W\|^{\frac{1}{n}}$.
(iii) $S_Z(W) < \frac{1}{\sigma(Z)}\|Z\|^{1-\frac{1}{n}}\|Z - W\|^{\frac{1}{n}}$.

Proof. We now prove (i).

Assume that $S_Z(W) = \frac{1}{D(Z)}\|Z\|^{1-\frac{1}{n}}\|Z - W\|^{\frac{1}{n}}$.

By (11) we have

$$S_Z(W)^n \leqslant \frac{1}{D(Z)^n}\left\|Z\begin{pmatrix}\delta v_1\\ -\gamma v_1\end{pmatrix}\right\|\left\|Z\begin{pmatrix}\delta v_2\\ -\gamma v_2\end{pmatrix}\right\|\cdots\left\|Z\begin{pmatrix}\delta v_n\\ -\gamma v_n\end{pmatrix}\right\|$$
$$= \frac{1}{D(Z)^n}\left\|(Z - W)\begin{pmatrix}\delta v_1\\ -\gamma v_1\end{pmatrix}\right\|\left\|Z\begin{pmatrix}\delta v_2\\ -\gamma v_2\end{pmatrix}\right\|\cdots\left\|Z\begin{pmatrix}\delta v_n\\ -\gamma v_n\end{pmatrix}\right\|$$
$$\leqslant \frac{1}{D(Z)^n}\|Z - W\|\|Z\|^{n-1}. \tag{13}$$

By using the same arguments as in the proof of Theorem 1, it follows from (13) that

(1) $\exists\alpha, \beta \in \mathbb{C}$, such that $Z = (\alpha U, \beta U)$ and $(-\bar{\beta}, \bar{\alpha})$ is an eigenvalue of W, where U is a unitary matrix.
(2) $\|Z\| = \|Z - W\|$.

Since $\| \ \|$ is a unitarily invariant norm, without loss of generality, by the generalized Schur decomposition [4], we assume that

$$W = \left(\begin{pmatrix} \gamma_1 & & * \\ & \ddots & \\ 0 & & -k\bar{\beta} \end{pmatrix}, \begin{pmatrix} \delta_1 & & * \\ & \ddots & \\ 0 & & k\bar{\alpha} \end{pmatrix}\right) \quad (k \neq 0) \tag{14}$$

and let $\hat{u}_n, e_n$ be the nth row of U and I respectively, where I is the unit matrix.
Then

$$\begin{aligned} |\alpha|^2 + |\beta|^2 = \|Z\|^2 &= \|Z - W\|^2 \\ &\geqslant (\alpha\hat{u}_n + k\bar{\beta}e_n)(\alpha\hat{u}_n + k\bar{\beta}e_n)^H + (\beta\hat{u}_n - k\bar{\alpha}e_n)(\beta\hat{u}_n - k\bar{\alpha}e_n)^H \\ &= |\alpha|^2 + |k\bar{\beta}|^2 + \bar{k}\alpha\beta\hat{u}_n e_n^H + k\bar{\alpha}\bar{\beta}e_n\hat{u}_n^H + |\beta|^2 + |k\bar{\alpha}|^2 - \bar{k}\alpha\beta\hat{u}_n e_n^H - k\bar{\alpha}\bar{\beta}e_n\hat{u}_n^H \\ &= |\alpha|^2 + |k\bar{\beta}|^2 + |\beta|^2 + |k\bar{\alpha}|^2. \end{aligned}$$

Thus

$$-k\bar{\beta} = k\bar{\alpha} = 0.$$

This contradicts the assumption that W is a regular matrix pair. So (i) holds. The proof for (ii) and (iii) is formally the same. $\square$

Now we cite other optimal bounds for $S_Z(W)$.

Theorem 2.6 of [1]. For $A \neq 0$, $B \neq 0$, $C \neq 0$, $D \neq 0$ the following are equivalent:

(i) $S_Z(W) = \frac{1}{D(Z)}\|(C^HC + D^HD)^{-1}\|^{\frac{1}{2}}(\|A\|\|D\| + \|B\|\|C\|)^{1-\frac{1}{n}}(\|D\|\|A - C\| + \|C\| \|B - D\|)^{\frac{1}{n}}$;
(ii) $S_Z(W) = \frac{1}{\gamma(Z)}\|(C^HC + D^HD)^{-1}\|^{\frac{1}{2}}(\|A\|\|D\| + \|B\|\|C\|)^{1-\frac{1}{n}}(\|D\|\|A - C\| + \|C\| \|B - D\|)^{\frac{1}{n}}$;
(iii) $S_Z(W) = \frac{1}{\sigma(Z)}\|(C^HC + D^HD)^{-1}\|^{\frac{1}{2}}(\|A\|\|D\| + \|B\|\|C\|)^{1-\frac{1}{n}}(\|D\|\|A - C\| + \|C\| \|B - D\|)^{\frac{1}{n}}$;
(iv) $\exists\varepsilon \in \mathbb{C}$, $|\varepsilon| = 1$ such that $A = \varepsilon\|A\|U$, $B = \|B\|U$, $C = -\varepsilon\|C\|V_C$, $D = \|D\|V_D$ where U, V_C, V_D are unitary matrices and $(-\varepsilon\|C\|, \|D\|)$ is an eigenvalue of W.

For $\alpha \neq 0$, $\beta \neq 0$, notice the fact that $\arg(-\frac{\bar{\beta}}{\bar{\alpha}}) = \arg\frac{\alpha}{\beta} + \pi$, where $\arg z$ is an argument of a complex number z. Then by Theorem 1 and Theorem 2.6 of [1] we have

Corollary 3. *For $Z \neq W$, $A \neq 0$, $B \neq 0$, the following are equivalent*:

(i) $S_Z(W) = \frac{1}{D(Z)}\|Z\|d_2(Z, W)^{\frac{1}{n}}$;
(ii) $S_Z(W) = \frac{1}{\gamma(Z)}\|Z\|d_2(Z, W)^{\frac{1}{n}}$;

(iii) $S_Z(W) = \frac{1}{\sigma(Z)}\|Z\|d_2(Z, W)^{\frac{1}{n}}$;
(iv) $\exists\alpha, \beta, \gamma, \delta \in \mathbb{C}$ *satisfying* $\arg\beta\gamma = \arg\alpha\delta + \pi$ *and* $|\alpha\gamma| = |\beta\delta|$ *such that* $Z = (\alpha U, \beta U)$ *and* (γ, δ) *is an eigenvalue of* W, *where* U *is a unitary matrix.*

Corollary 4. *For* $A \neq 0$, $B \neq 0$, $C \neq 0$, $D \neq 0$ *the following are equivalent*:

(i) $S_Z(W) = \frac{1}{D(Z)}\|(C^HC + D^HD)^{-1}\|^{\frac{1}{2}}(\|A\|\|D\| + \|B\|\|C\|)^{1-\frac{1}{n}}(\|D\|\|A - C\| + \|C\| \|B - D\|)^{\frac{1}{n}}$;
(ii) $S_Z(W) = \frac{1}{\gamma(Z)}\|(C^HC + D^HD)^{-1}\|^{\frac{1}{2}}(\|A\|\|D\| + \|B\|\|C\|)^{1-\frac{1}{n}}(\|D\|\|A - C\| + \|C\| \|B - D\|)^{\frac{1}{n}}$;
(iii) $S_Z(W) = \frac{1}{\sigma(Z)}\|(C^HC + D^HD)^{-1}\|^{\frac{1}{2}}(\|A\|\|D\| + \|B\|\|C\|)^{1-\frac{1}{n}}(\|D\|\|A - C\| + \|C\| \|B - D\|)^{\frac{1}{n}}$;
(iv) $\exists\alpha, \beta, \gamma, \delta \in \mathbb{C}$ *satisfying* $\arg\beta\gamma = \arg\alpha\delta + \pi$ *such that* $A = \alpha U, B = \beta U, C = \gamma V_C$, $D = \delta V_D$ *where* U, V_C, V_D *are unitary matrices and* (γ, δ) *is an eigenvalue of* W.

Thus Corollaries 3 and 4 show that Theorem 1 and Theorem 2.6 of [1] are not contradictory.

From the fact that

$$\|Z\|^2 = \|AA^H + BB^H\| \leqslant \|A\|^2 + \|B\|^2,$$

Theorem 2 implies that the inequalities (2.9), (3.3b) of [1], (5) of [2] (or (7) in this note) and (4.5), (4.11) of [3] are strict.

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